Definitions
Complex analysis qualifying course
MSU, Spring 2017

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This document was made as a way to study the material from the spring semester complex analysis qualifying course at Michigan State University, in spring of 2017. It serves as a companion document to the "Theorems" review sheet for the same class. The textbook for the course was *Complex Function Theory*, by Donald Sarason, and these notes closely follow that text.

Contents

1 Chapter 1: Complex Numbers

In this section, x, y denote real numbers.

Definition 1.1. Let $z = x + iy \in \mathbb{C}$. The **real part** of z is $\mathbf{Re}\,\boldsymbol{z} = x$.

Definition 1.2. Let $z = x + iy \in \mathbb{C}$. The **imaginary part** of z is $\mathbf{Im}\,\boldsymbol{z} = y$.

Definition 1.3. Let $z = x + iy \in \mathbb{C}$. The **modulus** of z, denoted $|z|$, is $\sqrt{x^2 + y^2}$.

Definition 1.4. Let $z = r(\cos\theta + i\sin\theta) \in \mathbb{C}$. Then θ is a **argument** of z, denoted $\theta = \mathbf{arg}\,\boldsymbol{z}$. The **principal argument** of z is the argument in the interval $(-\pi, \pi]$, denoted $\mathbf{Arg}\,\boldsymbol{z}$.

Definition 1.5. The **extended complex plane**, denoted $\overline{\mathbb{C}}$, is the space $\mathbb{C} \cup \{\infty\}$.

2 Chapter 2: Complex Differentiation

Definition 2.1. Let $u : \mathbb{R}^n \to \mathbb{R}$. The function u is **of class** $\boldsymbol{C^k}$ if the first k derivatives of u exist and are continuous.

Definition 2.2. Let $f : G \to \mathbb{C}$ where $G \subset \mathbb{C}$ is open. The function f is **differentiable** at $z_0 \in G$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. When it exists, it is called $f'(z_0)$.

Definition 2.3. Let $f : G \to \mathbb{C}$ where $G \subset \mathbb{C}$ is open. The function f is **holomorphic** on G if it is differentiable at every $z_0 \in G$.

Definition 2.4. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then f is called **entire**.

Definition 2.5. Let $f = u + iv$ be a complex valued function. The differential operators $\boldsymbol{\frac{\partial}{\partial x}}, \boldsymbol{\frac{\partial}{\partial y}}$ are defined by

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\partial}{\partial y} f = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Note that they are linear operators. In terms of these, we define the differential operators $\boldsymbol{\frac{\partial}{\partial z}}, \boldsymbol{\frac{\partial}{\partial \bar{z}}}$ by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \implies \frac{\partial}{\partial z} f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \implies \frac{\partial}{\partial \bar{z}} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Note that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are linear operators. Given this, we have the equivalent formulation,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$$
$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)$$

Definition 2.6. A **curve** in $\mathbb{C}$ is a continuous map $\gamma : I \to \mathbb{C}$, where I is any interval in $\mathbb{R}$.

Definition 2.7. A curve $\gamma : I \to \mathbb{C}$ is **differentiable at t_0** if the limit

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. When it exists, this limit is denoted $\boldsymbol{\gamma'(t_0)}$. If γ is differentiable at all $t_0 \in I$, then γ is **differentiable**. If γ is differentiable and $\gamma' : I \to \mathbb{C}$ is continuous, then γ is called $\boldsymbol{C^1}$.

Definition 2.8. A curve $\gamma : I \to \mathbb{C}$ is **regular at t_0** if it is differentiable and if $\gamma'(t_0) \neq 0$. If γ is C^1 and regular at every $t_0 \in I$, then γ is a **regular curve**.

Definition 2.9. Let $\gamma : I \to \mathbb{C}$ be a curve that is regular at t_0. The **direction** of γ at t_0 is $\arg \gamma'(t_0)$. We can also specify the direction with the unit tangent vector $\frac{\gamma'(t_0)}{|\gamma'(t_0)|}$.

Definition 2.10. Let γ_1, γ_2 be curves in $\mathbb{C}$ that intersect at $\gamma_1(t_1) = \gamma_2(t_2)$. The **angle between $\boldsymbol{\gamma_1}$ and $\boldsymbol{\gamma_2}$** is the angle $\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1) \pmod{2\pi}$.

Definition 2.11. Let f be a complex-valued function defined on an open set G, and let $z_0 \in G$. Then f is **conformal** at z_0 if for any curves γ_1, γ_2 such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and γ_j is regular at t_j, the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ is the same as the angle between γ_1 and γ_2.

Definition 2.12. A function $f : G \to \mathbb{C}$ is **harmonic** if is C^2 and satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

That is, f is in the kernel of the linear operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Definition 2.13. Let $u, v : G \to \mathbb{R}$ where $G \subset \mathbb{C}$ is open. The functions u, v are **harmonic conjugates** if they satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

3 Chapter 3: Linear Fractional Transformations

Definition 3.1. A **linear fractional transformation** is a function $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$, so that $ad - bc \neq 0$. (This rules out ϕ being constant.) If $c = 0$, we define $\phi(\infty) = \infty$, and if $c \neq 0$ then $\phi(\infty) = a/c$ and $\phi(-d/c) = \infty$.

Definition 3.2. Consider the space $\mathbb{C}^2$ under the equivalence relation $(z_1, z_2) \sim \lambda(z_1, z_2)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. The space $\mathbb{C}^2/\sim$ is **complex projective space**, denoted $\mathbb{CP}^1$.

Definition 3.3. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a non-singular matrix with complex entries, that is, $M \in \mathrm{GL}(2, \mathbb{C})$. The linear fractional transformation **induced by** M is the map $z \mapsto \frac{az+b}{cz+d}$.

Definition 3.4. Let z_1, z_2, z_3, z_4 be distinct points in $\overline{\mathbb{C}}$. The **cross ratio**, denoted $(z_1, z_2; z_3, z_4)$ is the image of z_4 under the unique linear fractional transformation ϕ so that $\phi(z_1) = \infty, \phi(z_2) = 0$, and $\phi(z_3) = 1$.

Definition 3.5. A **homothetic map** or **dilation** is a linear fractional transformation of the form $z \mapsto kz$ for some $k > 0$. It is induced by a matrix of the form

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

where $k > 0$.

Definition 3.6. A **rotation** is a linear fractional transformation of the form $z \mapsto \lambda z$ where $|\lambda| = 1$. It is induced by a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

where $|\lambda| = 1$.

Definition 3.7. A **translation** is a linear fractional transformation of the form $z \mapsto z + b$ where $b \in \mathbb{C}$. It is induced by a matrix of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where $b \in \mathbb{C}$.

Definition 3.8. The **inversion map** is the linear fractional transformation $z \mapsto \frac{1}{z}$. It is induced by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition 3.9. A **clirlce** is the image in $\overline{\mathbb{C}}$ of a circle on S^2 under the stereographic projection. Note that a clircle is either a circle in $\mathbb{C}^2$ or a line in $\mathbb{C}^2$ union the point at infinity.

4 Chapter 4: Elementary Functions

Definition 4.1. Let $z = x + iy$. The **complex exponential** function is defined by $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp z = e^z = e^x(\cos y + i \sin y)$.

Definition 4.2. The trigonometric and hyperbolic trigonometric functions are defined by

$$\begin{array}{ccc} \cos z = \dfrac{e^{iz} + e^{-iz}}{2} & \sin z = \dfrac{e^{iz} - e^{-iz}}{2} & \tan z = \dfrac{\sin z}{\cos z} \\ \sec z = \dfrac{1}{\cos z} & \csc z = \dfrac{1}{\sin z} & \cot z = \dfrac{1}{\tan z} \\ \cosh z = \dfrac{e^{z} + e^{-z}}{2} & \sinh z = \dfrac{e^{z} - e^{-z}}{2} & \tanh z = \dfrac{\sinh z}{\cosh z} \\ \cosh z = \dfrac{1}{\coth z} & \sinh z = \dfrac{1}{\sinh z} & \coth z = \dfrac{1}{\tanh z} \end{array}$$

Definition 4.3. Let $z \in \mathbb{C} \setminus \{0\}$. A **logarithm** of z is a complex number w so that $e^w = z$. (Note that there are infintely many such w for a given z.)

Definition 4.4. Let G be an open connected subset of $\mathbb{C}\setminus\{0\}$. A **branch of the argument** is a continuous function α such that $\alpha(z) = \arg z$ for $z \in G$.

Definition 4.5. Let G be an open connected subset of $\mathbb{C}\setminus\{0\}$. A **branch of the logarithm** is a continuous function ℓ so that $e^{\ell(z)} = z$ for $z \in G$.

Note: Given a set G, there may not exist a branch of arg or log.

Definition 4.6. The **principal branch of arg** is $\operatorname{Arg} z$, which exists on $\mathbb{C} \setminus (-\infty, 0]$.

Definition 4.7. The **principal branch of log** is $\operatorname{Log} z$, defined by $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$.

Definition 4.8. Let G be an open connected set of $\mathbb{C}$, and let f be a nonvanishing holomorphic function in G. A **branch of log f** is a continuous function $g : G \to \mathbb{C}$ so that $f(z) = e^{g(z)}$ for $z \in G$. (Note: A branch of log is the special case $f(z) = z$.)

Definition 4.9. Let f be holomorphic in G. The **logarithmic derivative** of f is $\frac{f'}{f}$.

Definition 4.10. Let G be an open connected subset of $\mathbb{C}$ and let f be a nonvanishing holomorphic function on G. Let $n \in \mathbb{N}$. A **branch of $f^{1/n}$** is a continuous function h in G so that $h(z)^n = f(z)$ for all $z \in G$.

Definition 4.11. Let $z, w \in \mathbb{C}$. We define the expression z^w to mean the set of values of $e^{w \log z}$.

5 Chapter 5: Power Series

Definition 5.1. A **infinite series** is a summation $\sum_{n=0}^{\infty} c_n$ with $c_n \in \mathbb{C}$.

Definition 5.2. The infinite series $\sum_{n=0}^{\infty} c_n$ **converges** if $\lim_{N\to\infty} \sum_{n=0}^{N} c_n$ converges and is finite. If it converges, this limit is the **sum** of the series.

Definition 5.3. The series $\sum_{n=0}^{\infty} c_n$ **converges absolutely** if $\sum_{n=0}^{\infty} |c_n|$ converges.

Definition 5.4. Let g_n be a sequence of complex-valued functions defined in G. The sequence **converges** (pointwise) if $\lim_{n\to\infty} g_n(z)$ exists and is finite for each $z \in G$.

Definition 5.5. Let g_n be a sequence of complex-valued functions defined in G with pointwise limit g. The sequence **converges uniformly** to g on S if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that

$$n \geq N \implies |g(z) - g_n(z)| < \epsilon, \quad \forall z \in S$$

Definition 5.6. Let g_n be a sequence of complex-valued functions defined in G. The sequence is **uniformly Cauchy** on S if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$n, m \geq N \implies |g_n(z) - g_m(z)| < \epsilon, \quad \forall z \in S$$

Definition 5.7. Let g_n be a sequence of complex-valued functions defined in G. The sequence is **converges locally uniformly** in G if each point in G has an open neighborhood in which the sequence converges uniformly. Equivantly, it converges locally uniformly if it converges uniformly on each compact subset of G.

Definition 5.8. Let f_n be a sequence of complex-valued functions. The series $\sum_{k=0}^{\infty} f_n$ **converges** if the sequence of partial sums converges. It **converges unformly** if the sequence of partial sums converges uniformly. It converges **locally uniformly** if the sequence of partial sums converges locally uniformly.

Definition 5.9. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ where z_0, a_i are complex constants. If a power series converges to a function f on a set G, then the series **represents** f on G.

Definition 5.10. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. The **radius of convergence** for the series the supremum over all R so that the series converges on the disk $|z - z_0| < R$.

Definition 5.11. Let a_n be a sequence of real numbers. The **lim sup** and **lim inf** of the sequence are

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup\{a_k : k \geq n\}\right)$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf\{a_k : k \leq n\}\right)$$

Note that these limits always exist, since the sequence of suprema/infima are decreasing/increasing sequences respectively.

Definition 5.12. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ be power series with the same center. The **Cauchy product** is the power series

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n$$

6 Chapter 6: Complex Integration

Definition 6.1. Let $[a,b] \subset \mathbb{R}$. A function $\phi : [a,b] \to \mathbb{C}$ is **piecewise continuous** if it is continuous at all but finitely many points of $[a,b]$ and has finite one-sided limits at each discontinuity.

Definition 6.2. Let $\phi : [a,b] \to \mathbb{C}$ be piecewise continuous. Then the integrals

$$\int_a^b \operatorname{Re}\phi(t)dt \qquad \int_a^b \operatorname{Im}\phi(t)dt$$

are defined as usual Riemann integrals of real functions. The **integral of ϕ over $[a,b]$** is defined by

$$\int_a^b \phi(t)dt = \int_a^b \operatorname{Re}\phi(t)dt + i\int_a^b \operatorname{Im}\phi(t)dt$$

Definition 6.3. A function $\phi : [a,b] \to \mathbb{C}$ is **differentiable** at $t_0 \in [a,b]$ if $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ are differentiable at t_0. If ϕ is differentiable at t_0, then its derivative is defined to be

$$\phi'(t_0) = (\operatorname{Re}\phi)'(t_0) + i(\operatorname{Im}\phi)'(t_0)$$

Definition 6.4. A function $\phi : [a,b] \to \mathbb{C}$ is **piecewise C^1** if it is continuous, differentiable at all but finitely many points, has a continuous derivative where the derivative exists, and the derivative has finite one-sided limits at its discontinuities.

Definition 6.5. Let $\gamma : [a,b] \to C$ be a piecewise C^1 curve. A **reparametrization** of γ is a curve $\gamma_1 = \gamma \circ \beta$ where $\beta : [c,d] \to [a,b]$ is strictly increasing, piecewise C^1, and surjective.

Definition 6.6. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 curve. The **length** of γ is defined as

$$\boldsymbol{L(\gamma)} = \int_a^b |\gamma'(t)|dt$$

(Note: This is not an intrinsic geometric definition, since it appears to depend on the parametrization of γ. However, one can show that it does not depend on the parametrization.)

Definition 6.7. Let $G \subset \mathbb{C}$, and $f : G \to \mathbb{C}$. Let $\gamma : [a,b] \to G$ be a piecewise C^1 curve. The **integral of f over γ** is the integral

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Definition 6.8. Let $z_1, z_2 \in \mathbb{C}$. Then $\boldsymbol{[z_1, z_2]}$ is the line segment with endpoints z_1, z_2, directed from z_1 to z_2. One common parametrization of this is $\gamma : [0,1] \to [z_1, z_2]$ given by $\gamma(t) = (1-t)z_1 + tz_2$.

Definition 6.9. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. The **reverse** of γ is the curve $-\gamma : [-b,-a] \to \mathbb{C}$ defined by $(-\gamma)(t) = \gamma(-t)$. This reverses the direction in which γ traverses the image curve.

7 Chapter 7: Core Versions of Cauchy's Theorem

Definition 7.1. Let $z_1, z_2, z_3 \in \mathbb{C}$. The **triangle** $T(z_1, z_2, z_3)$ is the set $[z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_1]$.

Definition 7.2. A subset of $\mathbb{C}$ is **convex** if for every $z_1, z_2 \in \mathbb{C}$ the line segment $[z_1, z_2]$ is contained in $\mathbb{C}$.

Definition 7.3. A subset G of $\mathbb{C}$ is **star shaped** if there exists $z_0 \in G$ so that $[z_0, z] \subset G$ for every $z \in G$. (Note that every convex set is star shaped, but the converse is false.)

Definition 7.4. Let $f : G \to \mathbb{C}$ be holomorphic. A **primitive** of f is a holomorphic function $F : G \to \mathbb{C}$ such that $F' = f$.

Definition 7.5. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve and let $\phi : \operatorname{im}\gamma \to \mathbb{C}$ be continuous. The **Cauchy integral** of ϕ over γ is the function $f : \mathbb{C} \setminus \operatorname{im}\gamma \to \mathbb{C}$ defined by

$$f(z) = \int_\gamma \frac{\phi(w)}{w - z} dw$$

Definition 7.6. Let $f : G \to \mathbb{C}$ be holomorphic, and let $z_0 \in G$ so that $f(z_0) = 0$. The point z_0 is a **zero of order m** if $f^{(n)}(z_0) = 0$ for $n = 0, \ldots, m-1$ and $f^{(m)} \neq 0$.

8 Laurent Series and Isolated Singularities

Note: Prof Schenker presented this material in a different order in class, giving a different definition for singularities, but it all turns out to be logically equivalent.

Definition 8.1. A **Laurent series** is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

The series is defined to converge if both the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

converge. When it converges, the sum of the series is

$$\lim_{N \to \infty} \sum_{n=-N}^{N} a_n (z - z_0)^n$$

The series

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

is the **principal part** of the Laurent series.

Definition 8.2. A **punctured disk** centered at z_0 is an open annulus $0 < |z - z_0| < R$.

Definition 8.3. Let f be holomorphic in G. A point $z_0 \in G$ is an **isolated singularity** of f if $z_0 \notin G$ but G contains a punctured disk centered at z_0.

Definition 8.4. Let f be holomorphic in G with an isolated singularity at z_0, and let $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be a Laurent series for f centered at z_0.

1. z_0 is a **removable singularity** if $a_n = 0$ for all $n < 0$. In this case, f can be extended to a holomorphic function on $G \cup \{z_0\}$ by defining $f(z_0) = a_0$.

2. z_0 is a **pole of order m** for some $m \in \mathbb{N}$ if $a_{-m} \neq 0$ but $a_n = 0$ for $n < -m$. That is, the principal part of the Laurent series is eventually zero, and hence forms a rational function.

3. z_0 is an **essential singularity** if it is not one of the above. That is, the principal part of the Laurent series has infinitely many nonzero terms.

Definition 8.5. Let f be holomorphic with an isolated singularity at z_0. The **residue** of f at z_0, denoted $\mathbf{res}_{z_0} f$, is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of f near z_0.

9 Cauchy's Theorem

Definition 9.1. Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise C^1, and let $f : \gamma([a, b]) \to \mathbb{C} \setminus \{0\}$ be continuous. We know there exists a continuous $\psi : [a, b] \to \mathbb{C} \setminus \{0\}$ so that $f \circ \gamma = e^{\psi}$. The **increment in $\log f$ on γ**, denoted $\Delta(\log f, \gamma)$ is $\psi(b) - \psi(a)$. The **increment in $\arg f$ on γ**, denoted $\Delta(\arg f, \gamma)$, is $\mathrm{Im}(\Delta(\log f, \gamma)$. (Note that if γ is a closed curve, then $\Delta(\arg f, \gamma) = -i\Delta(\log f, \gamma)$.

Definition 9.2. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve, and let z_0 be a point not in the trace of γ. The **winding number** of γ around z_0 is $\frac{1}{2\pi}\Delta(\arg(z - z_0)\gamma)$. This is also called the **index of z_0 with respect to γ**, and denoted $\mathrm{ind}_\gamma(z_0)$.

Definition 9.3. A **contour** is a formal sum

$$\Gamma = \sum_{j=1}^{p} n_j \gamma_j$$

where γ_j are piecewise C^1 closed curves and n_j are integers. We can think of any curve γ as a contour 1γ.

Definition 9.4. Let $\Gamma = \sum_j n_j \gamma_j$ be a contour and let f be a continuous complex valued function defined on each γ_j. Then we define the integral over Γ by

$$\int_\Gamma f(z)dz = \sum_{j=1}^{p} n_j \int_{\gamma_j} f(z)dz$$

Definition 9.5. We define an equivalence relation on the set of contours by $\Gamma \sim \Gamma'$ if for every continuous function f,

$$\int_\Gamma f(z)dz = \int_{\Gamma'} f(z)dz$$

We do not distinguish between contours that are equivalent in this way.

Definition 9.6. Let $\Gamma = \sum_j n_j \gamma_j$ and $\Gamma' = \sum_j n'_j \gamma_j$ be contours. The **sum** is

$$\Gamma + \Gamma' = \sum_{j=1}^{p} (n_j + n'_j)\gamma_j$$

This binary operation gives an abelian group structure to the set of equivalence classes of contours.

Definition 9.7. Let $\Gamma = \sum_j n_j \gamma_j$ be a contour and let $z_0 \in \mathbb{C}$ be a point not on Γ. The **winding number** of Γ around z_0 is

$$\operatorname{ind}_\Gamma(z_0) = \sum_{j=1}^{p} n_j \operatorname{ind}_{\gamma_j}(z_0)$$

Definition 9.8. Let $\Gamma = \sum_j n_j \gamma_j$ be a contour. The **interior** of Γ is the set

$$\{z \in \mathbb{C} \setminus \Gamma : \operatorname{ind}_\Gamma(z) \neq 0\}$$

The **exterior** of Γ is the set

$$\{z \in \mathbb{C} \setminus \Gamma : \operatorname{ind}_\Gamma(z) = 0\}$$

Note that both the interior and exterior of Γ are open sets. Also note that the interior is bounded, and the exterior is unbounded. Also, the boundary points of the interior and exterior lie in Γ.

Definition 9.9. A contour Γ is **simple** if $\operatorname{ind}_\Gamma(z)$ is zero or one for every $z \in \mathbb{C} \setminus \Gamma$.

Definition 9.10. Let $G \subset \mathbb{C}$ be open. Two closed curves $\gamma_0, \gamma_1 : [0,1] \to G$ are **homotopic** if there is a continuous map $\gamma : [0,1] \times [0,1] \to G$ so that $\gamma(t,0) = \gamma_0(t), \gamma(t,1) = \gamma_1(t)$, and $\gamma(0,s) = \gamma(1,s)$ for all s, t.

10 Riemann Mapping Theorem

Definition 10.1. A **domain** is a nonempty connected open subset of $\mathbb{C}$.

Definition 10.2. A domain $G \subset \mathbb{C}$ is **simply connected** if $\overline{\mathbb{C}} \setminus G$ is connected. (Note: This is equivalent to usual topological simple connectedness.)

Definition 10.3. A **univalent** holomorphic function is injective.

Definition 10.4. Two domains $G_1, G_2 \subset \mathbb{C}$ are **conformally equivalent** if there is a univalent holomorphic function $f : G_1 \to \mathbb{C}$ such that $f(G_1) = G_2$. (Note: This is an equivalence relation on domains in $\mathbb{C}$.)

Definition 10.5. Let X be a topological space. Then $\boldsymbol{C(X)}$ is the space of continuous functions $f : X \to \mathbb{R}$.

Definition 10.6. Let X be a topological space. A family of functions $F \subset C(X)$ is **equicontinuous** if for every $x \in X$ and $\epsilon > 0$, there exists a neighborhood U_x of x such that

$$y \in U_x \text{ and } f \in F \implies |f(x) - f(y)| < \epsilon$$

Definition 10.7. Let X be a topological space. A family of functions $F \subset C(X)$ is **pointwise bounded** if for every $x \in X$,

$$\sup_{f \in F}\{|f(x)|\} < \infty$$

Definition 10.8. A family $\{f_i : G \to \mathbb{C}\}_{i \in I}$ is **uniformly bounded** if there exists $M \in \mathbb{R}$ so that

$$|f_i(z)| \leq M$$

for all $i \in I$ and all $z \in G$.

Definition 10.9. A family $\{f_i : G \to \mathbb{C}\}_{i \in I}$ of functions is **locally uniformly bounded** if each point $z \in G$ has a neighborhood in which the family is uniformly bounded. Equivalently, the family is locally uniformly bounded if it is uniformly bounded on each compact subset of the domain.

Definition 10.10. A family $\{f_i : G \to \mathbb{C}\}_{i \in I}$ of functions is **normal** if every sequence from the family has a locally uniformly convergent subsequence.